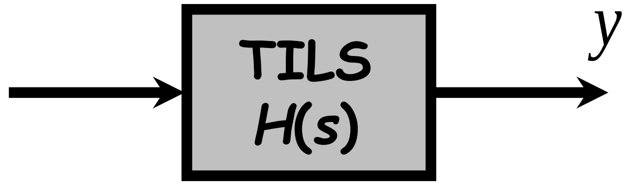
Signal filtering

- Signal filtering
- TILS ≡ RLC filters
- Voltage – current relationships for RLC elements
- Harmonic excitation
- Amplitude- and phase-frequency characteristics
- Ideal filters – a taxonomy
- Signal filtering – Fourier series
- Signal filtering – Fourier transform and convolution
- How does filtration alter frequency characteristics?
- Signal filtering by convolution in time domain
- Summary

Signal filtering

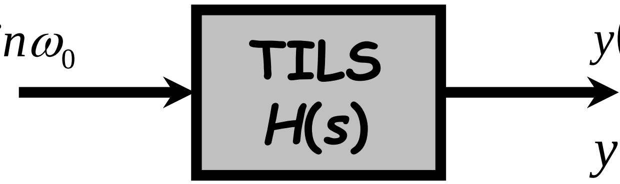
e^{st}



$$y(t) = H(s)e^{st} \longrightarrow \boxed{H(s) = \frac{y(t) \leftarrow e^{st}}{x(t) = e^{st}}}$$

$$x(t) = \sum_n X_n e^{jn\omega_0 t}, s = jn\omega_0$$

$$x(t) \leftrightarrow X(j\omega), s = j\omega$$



$$y(t) = \sum_n X_n H(jn\omega_0) e^{jn\omega_0 t}$$

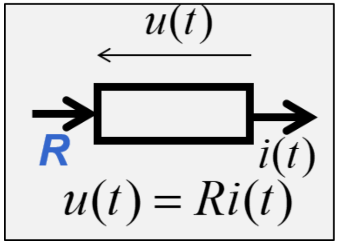
$$y(t) \leftrightarrow Y(j\omega) = H(j\omega) X(j\omega)$$

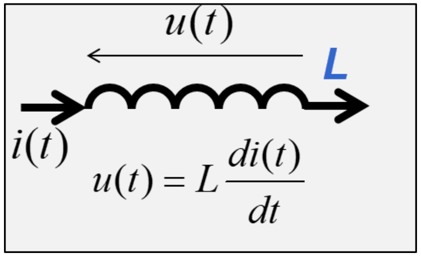
Signal processing in a TILS (RLC networks) needs determination of a transfer function *H*(*s*) which depends on an internal structure of an RLC network.

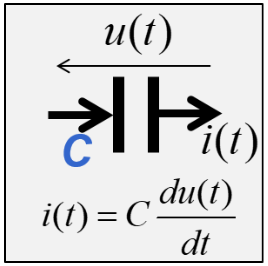
<u>Generic approach</u> to determine the transfer function *H*(*s*) is to find an ordinary differential equation (ODE) describing an RLC network (output *y*(*t*) in terms of input *x*(*t*)), and then „solving" the ODE using a substitution $x(t) = e^{st}$ oraz $y(t) = H(s)e^{st}$.

<u>Engineering approach</u> to determine the transfer function *H*(*s*) deploys the *„s"* (Laplace transform) approach.

Voltage-current relationships for elements RLC







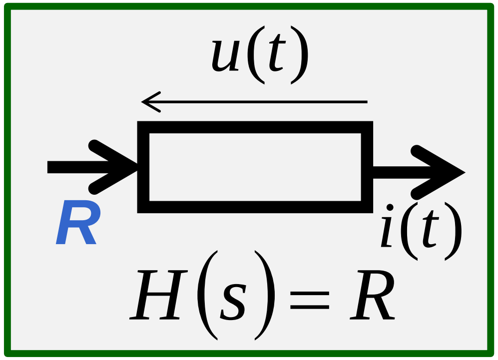
Transfer function *H*(*s*) for RLC elements („*s*" equivalents)

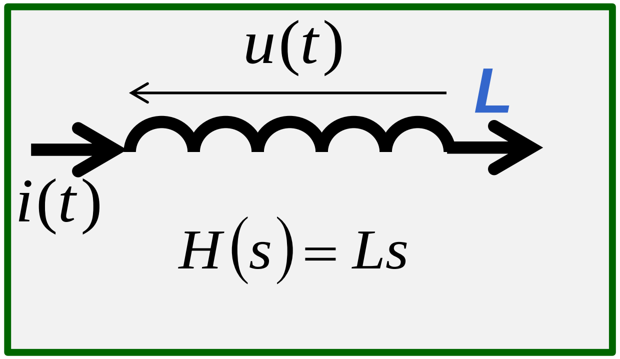
$$H(s) = \frac{y(t) \leftarrow e^{st}}{e^{st}}$$

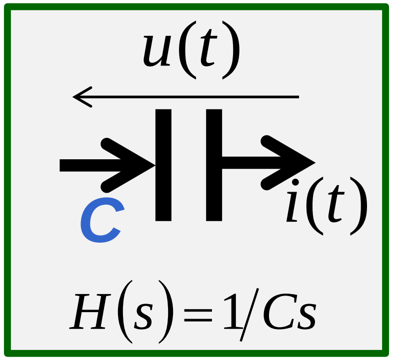
$$e^{st} \rightarrow y(t) = H(s)e^{st}$$

$$= \frac{\text{response (voltage) to exponential excitation}}{\text{exponential excitation (current)}}$$

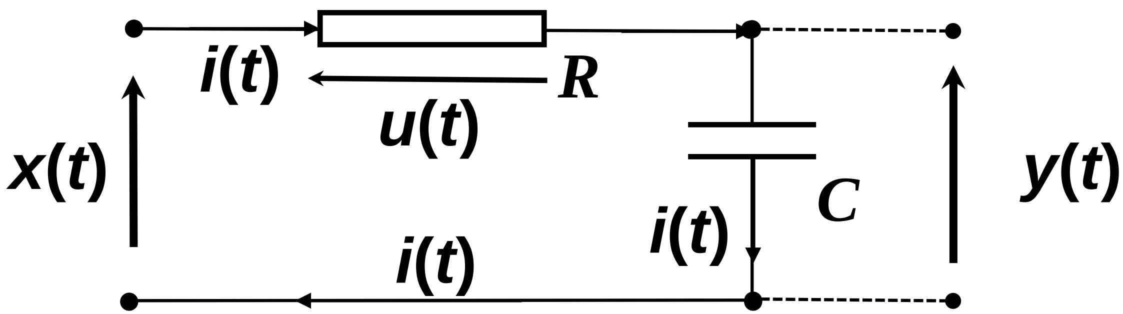
WORK IN PROGRESS







Lowpass RC filter - generic approach



$$\text{(VKL)} \quad x = u + y$$

$$\text{(VC - R)} \quad u = iR$$

$$\text{(VC - C)} \quad i = C\, dy/dt$$

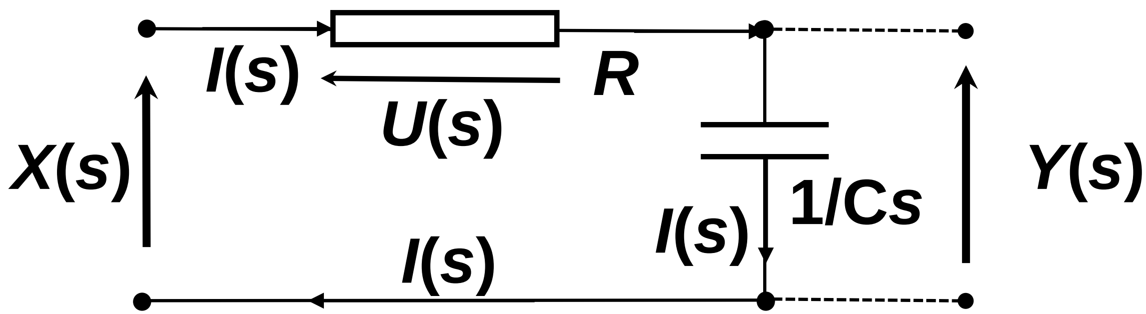
$$\frac{x-y}{R} = C\frac{dy}{dt}$$

$$x = y + 1/\omega_\mathrm{b} \times \frac{dy}{dt}, RC = 1/\omega_\mathrm{b}$$

$$x = e^{st} \quad y = H(s)e^{st}$$

$$H(s) = \frac{1}{1 + s/\omega_\mathrm{b}}$$

Lowpass RC filter
- „s” (Laplace transform) approach



1. Replace RLC elements by their transfer functions („*s*” equivalents).

2. Replace all currents and voltages by their „*s*” equivalents and change the notation to a capital font.

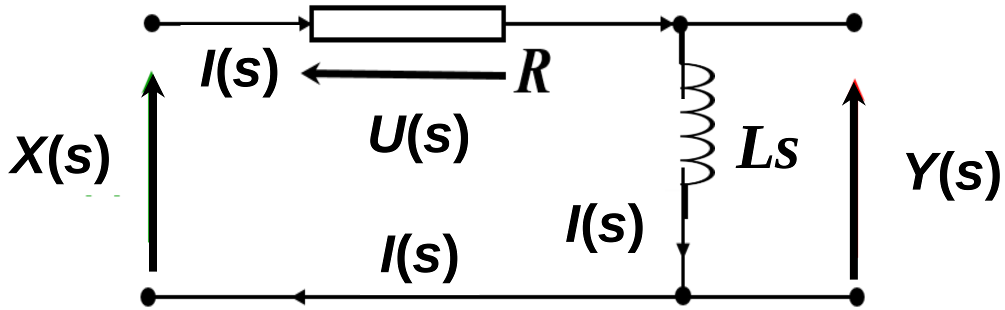
3. Arrange a set of all equations following the voltage and current Kirchhoff laws.

4. Solve the set for the output signal $Y(s)$ in terms of the input signal $X(s)$.

5. The transfer function is equal to $H(s) = Y(s)/X(s)$.

Lowpass RL filter
- „s” (Laplace transform) approach

$$X(s) = U(s) + Y(s)$$
$$U(s) = I(s)R$$
$$Y(s) = LsI(s)$$



$$H(s) = Y(s)/X(s)$$
$$H(s) = \frac{\tau s}{1+\tau s} \qquad \tau = L/R$$

Harmonic excitation

$$x(t) = \cos\omega t = \\ = \tfrac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$y(t) = \tfrac{1}{2} H(s := j\omega) e^{j\omega t} + \\ + \tfrac{1}{2} H(s := -j\omega) e^{-j\omega t}$$

$$\longrightarrow \boxed{H(s := j\omega)} \longrightarrow$$

$$H(s := j\omega) = \underbrace{\left|H(j\omega)\right|}_{A(\omega)} \exp[\, j\, \underbrace{\varphi(\omega)}_{\arg H(j\omega)}\,]$$

$$A(\omega) = A(-\omega)$$

a - f function

$$\varphi(\omega) = -\varphi(-\omega)$$

p - f function

$$y(t) = A(\omega)\cos[\omega t + \varphi(\omega)]$$

WORK IN PROGRESS

Filtering does not change frequency of a harmonic excitation; does change amplitude and phase in relation to frequency.

Amplitude- and phase-frequency characteristics

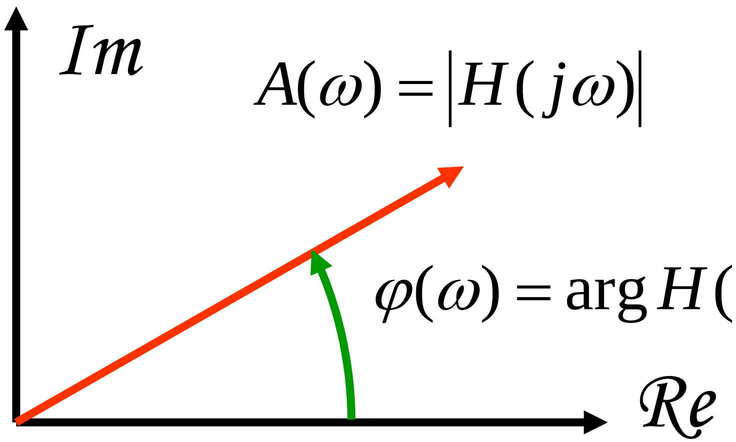
Transfer function *H(jω)*

$$H(j\omega)\in C$$

$$H(j\omega)=\underbrace{|H(j\omega)|}_{A(\omega)}\exp[j\ \underbrace{\varphi(\omega)}_{\arg H(j\omega)}\ ]$$

$$A(\omega)=A(-\omega)$$

$$\varphi(\omega)=-\varphi(-\omega)$$



$$A(\omega)=|H(j\omega)|$$

$$\varphi(\omega)=\arg H(j\omega)$$

Transfer function *H(jω)*
Exponential (polar) form

The a-f characteristic of a filter $A(\omega)=1$ for all frequencies ω.

The p-f characteristic of a filter is:

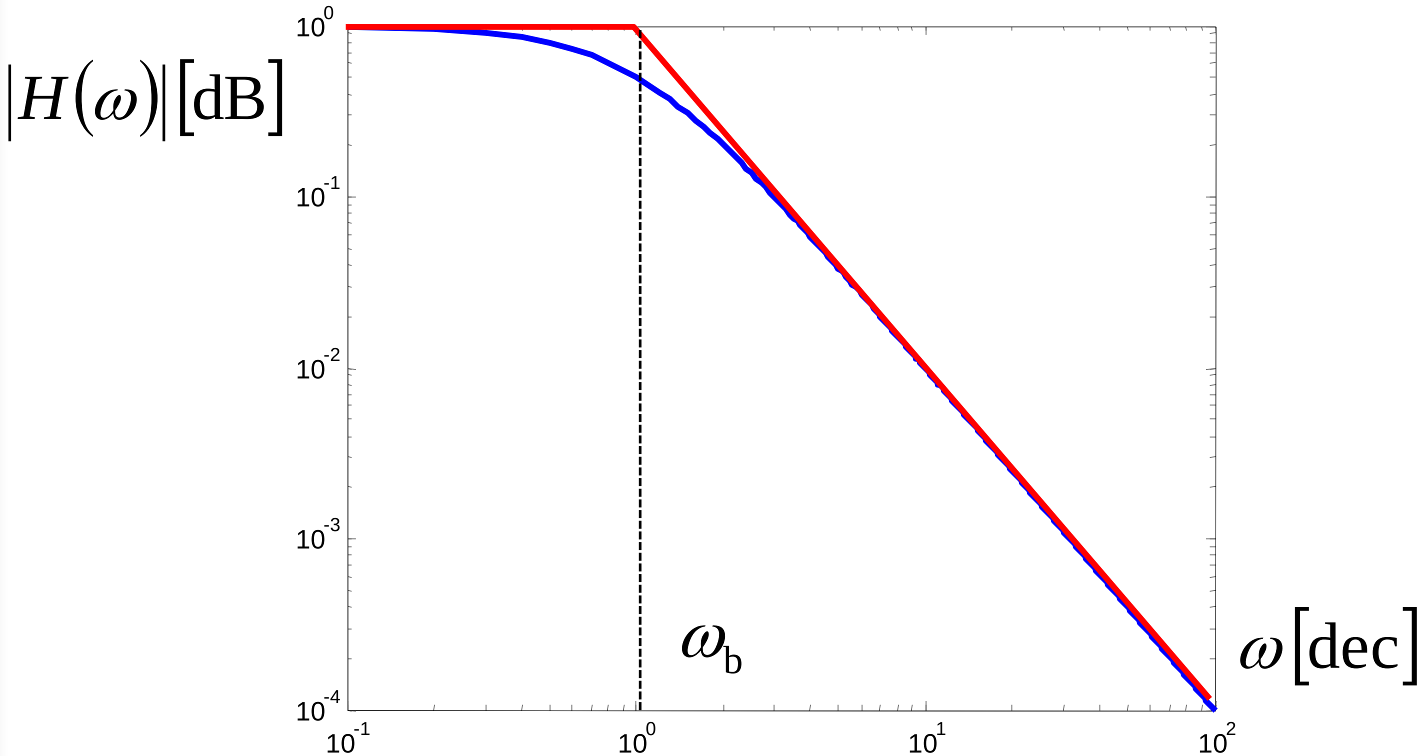
$$\varphi(\omega)=\begin{cases}+\pi/2,\omega>0\\-\pi/2,\omega>0\end{cases}.$$

Determine and sketch a transfer function $H(\omega)$ of the filter.

Lowpass filter RC a-f characteristic

$$H(j\omega)=\frac{1}{1+j\omega/\omega_b},\quad |H(j\omega)|=\frac{1}{\sqrt{1+(\omega/\omega_b)^2}}$$

$$20\log|H(j\omega)|=\begin{cases}0, & \omega/\omega_b \ll 1\\ -20\log(\omega/\omega_b), & \omega/\omega_b \gg 1\end{cases}$$

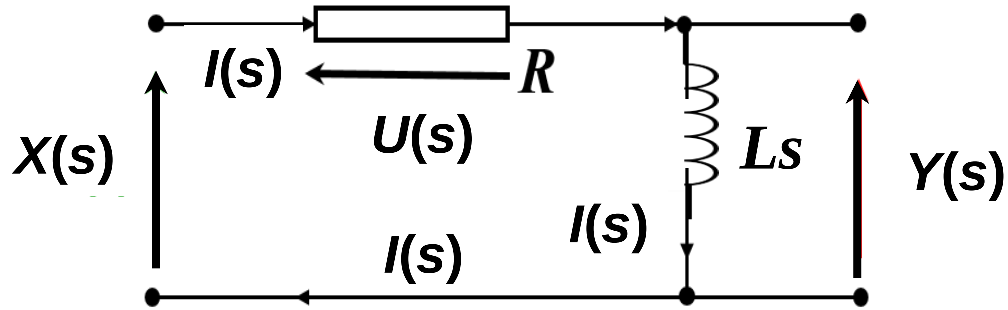


log-log amplitude characteristics of the LPF (1st order) (Bode plot)

Lowpass filter RL
a-f characteristic

$$H(s)=Y(s)/X(s)$$

$$H(s)=\frac{\tau s}{1+\tau s} \qquad \tau=L/R$$



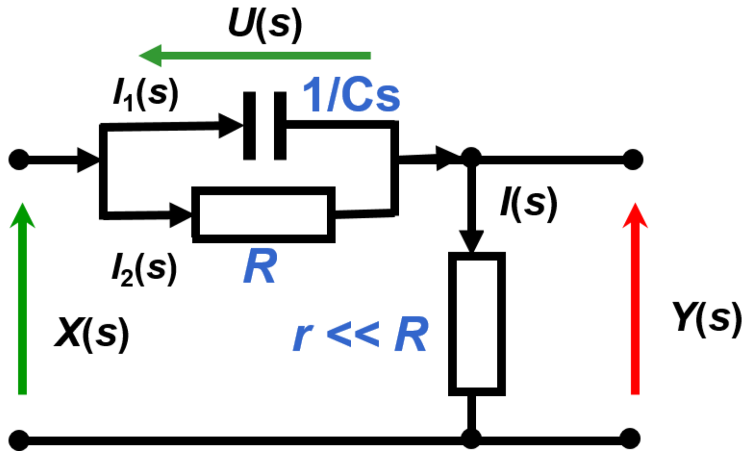
Find log-log amplitude characteristics of the RL filter.

Preemphasis filter

$$Y = rCsU + \frac{r}{R}U$$

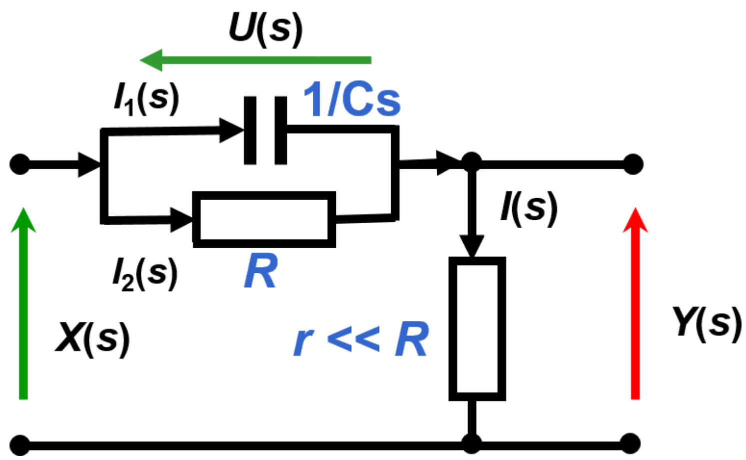
$$U = X - Y$$

$$Y = rCs(X-Y) + \frac{r}{R}(X-Y)$$



$$H(s) = Y(s)/X(s) \cong \frac{r}{R} \times \frac{1+RCs}{1+rCs}$$

Preemphasis filter – a-f characteristic



$$r \ll R$$

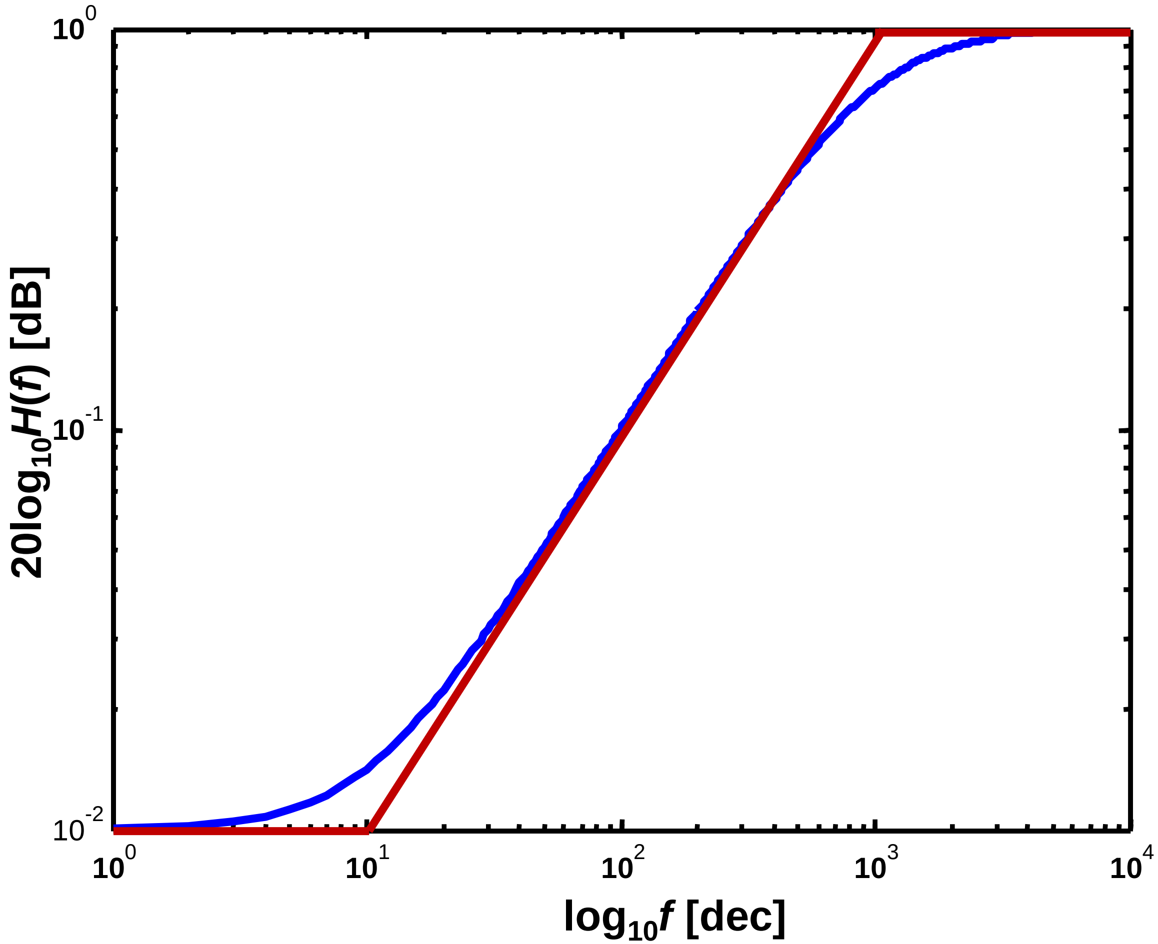
$$\omega_1 = \frac{1}{RC}, \omega_2 = \frac{1}{rC}$$

$$\omega_1 \ll \omega_2$$

$$H(j\omega) = \frac{r}{R} \times \frac{1 + sRC}{1 + srC}\bigg|_{s=j\omega} = \frac{r}{R} \times \frac{1 + j\omega/\omega_1}{1 + j\omega/\omega_2}$$

$$A(\omega) = |H(j\omega)| = \frac{r}{R} \times \frac{\sqrt{1 + (\omega/\omega_1)^2}}{\sqrt{1 + (\omega/\omega_2)^2}} = \begin{cases} r/R, \omega \ll \omega_1 \\ \omega/\omega_2, \omega_1 \ll \omega \ll \omega_2 \\ 1, \omega \gg \omega_2 \end{cases}$$

Preemphasis filter – a-f characteristic



Preemphasis filter
A-f characteristic in a log-log (dec-dB) scale

decibel [dB] & decade [dec]

Two frequencies f_1, f_2 are spaced by 1 dec if $f_1/f_2 = 10$ so f [Hz] $\rightarrow \log_{10} f$ [dec]

$f_2 = 1$ MHz, $f_1 = 10$ MHz

Two signals with powers P_1, P_2 are spaced by 1 Bel (or 10 deciBels) if $P_1/P_2 = 10$.

$P_1/P_2 = 10 \rightarrow \log_{10} P_1/P_2 = 1$ B or $10\log_{10} P_1/P_2 = 10$ dB

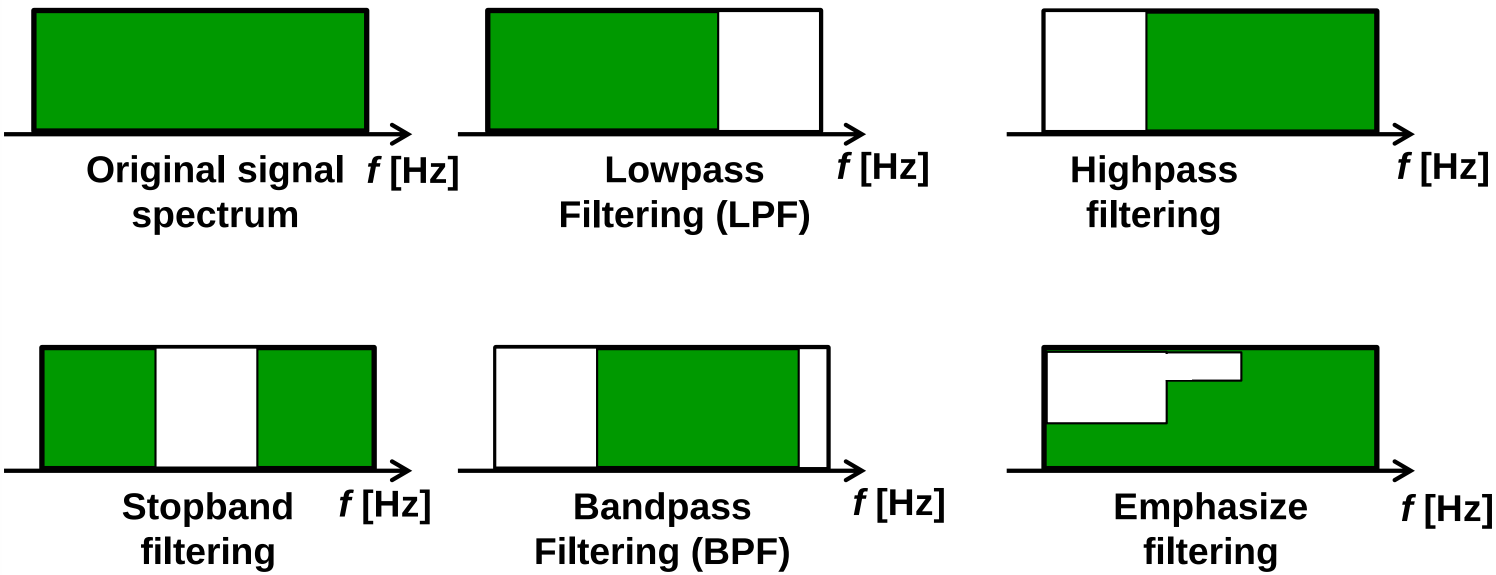
$10\log_{10} S/N$ = SNR [dB]
Signal to Noise Ratio

Comparing voltages U_1, U_2 needs the formula $20\log_{10} U_1/U_2$.

Ideal filters – a taxonomy

RLC filters are analog circuits which perform signal spectrum shaping, specifically:
- to remove unwanted frequency components or/and
- to emphasize wanted frequency components.

Types of <u>ideal</u> filters *H(f)*:



Two important properties of ideal filters:
- brickwall shaped edges
- full stopband
<u>These properties are not attainable with R, L, C elements according to the Paley-Wiener condition.</u>

Signal filtering – Fourier series

$$x(t)=\sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}$$

$$0\le t\le T,\ \omega_0=2\pi/T$$

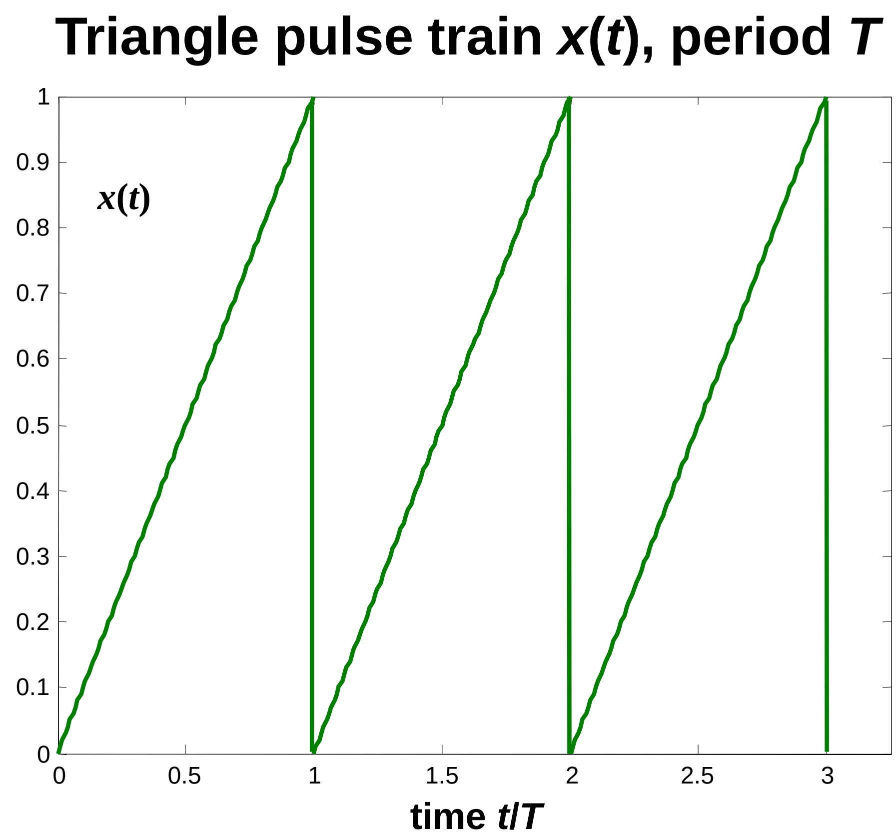
$$x(t) \longrightarrow \boxed{H(s:=j\omega)} \longrightarrow y(t)=?$$

$$X_n e^{jn\omega_0 t} \qquad\qquad X_n H(jn\omega_0) e^{jn\omega_0 t}$$

The exponential Fourier series of the output signal *y*(*t*)

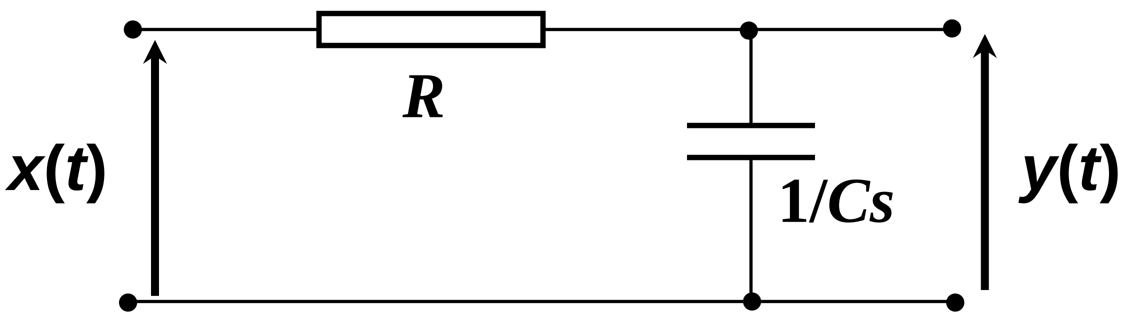
$$y(t)=\sum_{n=-\infty}^{+\infty} \underbrace{X_n H(jn\omega_0)}_{Y_n} e^{jn\omega_0 t}=\sum_{n=-\infty}^{+\infty} Y_n e^{jn\omega_0 t}$$

Signal filtering - example



$$x(t) = \frac{1}{2} + \frac{j}{2\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{n} e^{jn\omega_0 t}, \quad \omega_0 = 2\pi/T$$

Lowpass filter (integrator)



$$H(s) = \frac{1/Cs}{R + 1/Cs} = \frac{1}{1 + RCs}$$

$$H(j\omega) = \frac{1}{1 + jRC\omega} = \frac{1}{1 + j\omega/\omega_b}$$

$$1/\omega_b = RC$$

Input/output signals – Fourier series

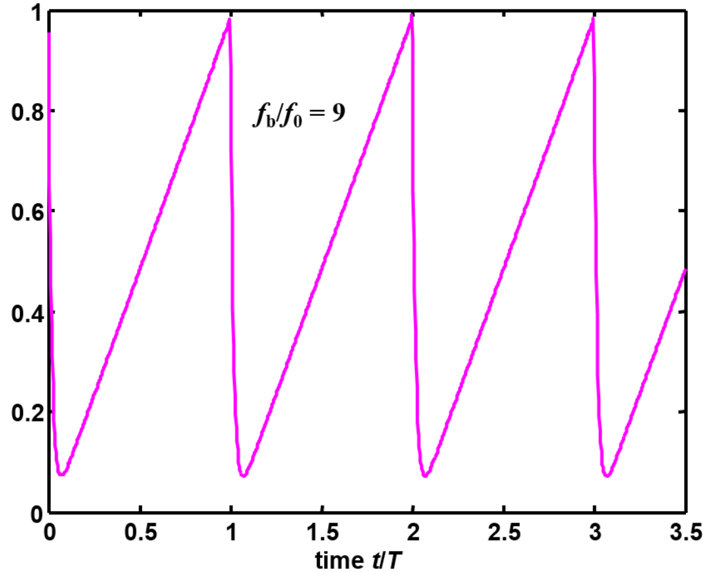
$$x(t)=\frac{1}{2}+\frac{j}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{1}{n}e^{jn\omega_0 t}$$

$$H(j\omega)=\frac{1}{1+j\omega/\omega_\mathrm{b}}$$

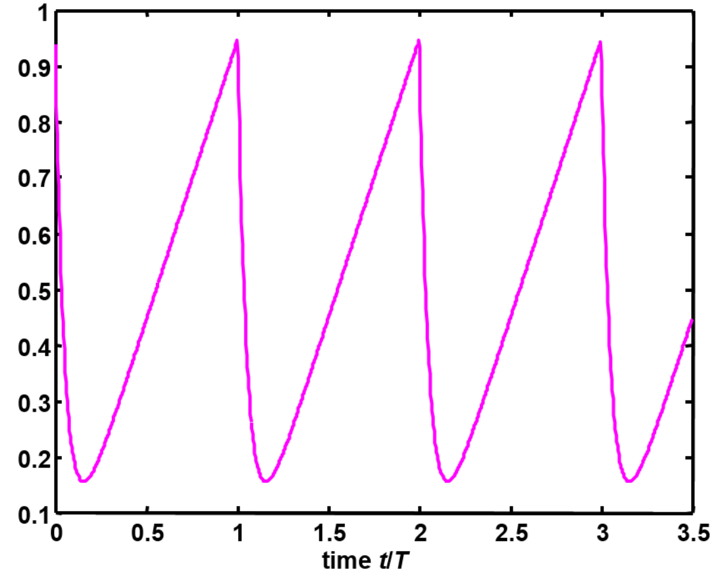
$$y(t)=\sum_{n=-\infty}^{+\infty}\underbrace{X_n H(jn\omega_0)}_{Y_n}e^{jn\omega_0 t}$$

$$y(t)=\frac{1}{2}+\frac{j}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{1}{n}\times\frac{1}{1+jn\omega_0/\omega_\mathrm{b}}e^{jn\omega_0 t}$$

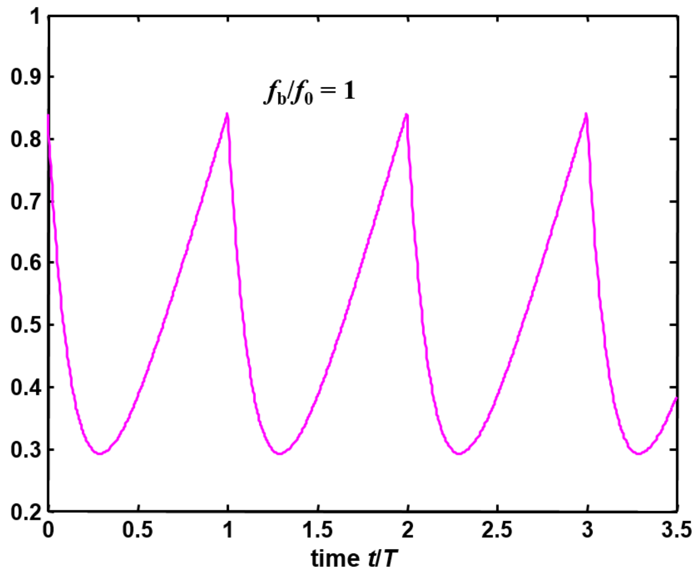
Output signal $y(t)$ (f_b/f_0 = 9)



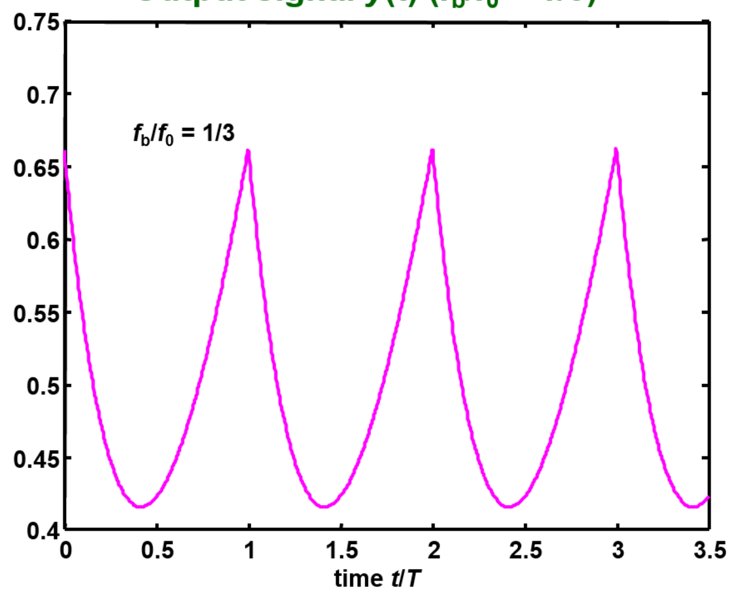
Output signal $y(t)$ (f_b/f_0 = 3)



Output signal $y(t)$ (f_b/f_0 = 1)



Output signal $y(t)$ (f_b/f_0 = 1/3)



Signal filtering by transfer function

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$x(t) \longrightarrow \boxed{H(s:=j\omega)} \longrightarrow y(t)$$

$$\frac{1}{2\pi}X(j\omega)e^{j\omega t} \longrightarrow \frac{1}{2\pi}H(j\omega)X(j\omega)e^{j\omega t}$$

Fourier transform of an output signal *y*(*t*)

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t}d\omega$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

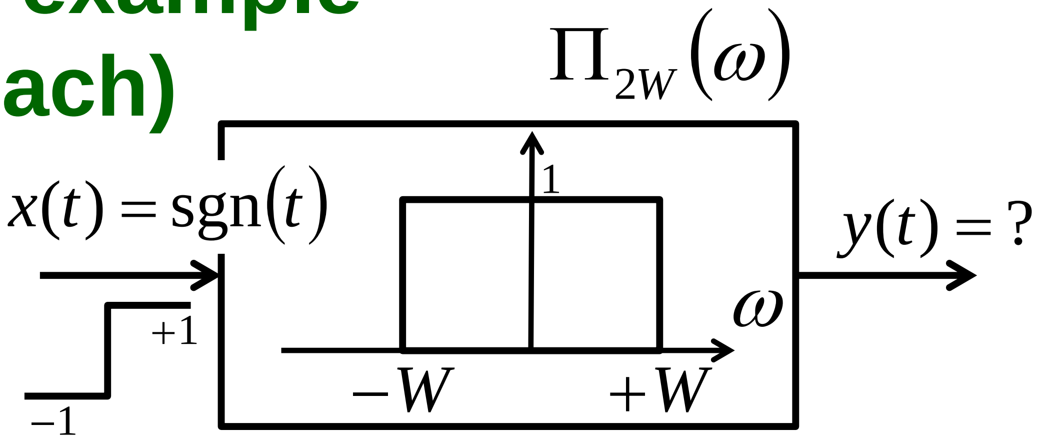
Fourier transform $Y(j\omega)$ of an output signal of a filter $y(t)$ is equal to a product of Fourier transforms $H(j\omega)\,X(j\omega)$ where $H(j\omega)$ is a transfer function of a filter and $x(t) \leftrightarrow X(j\omega)$.

The final step is the inverse Fourier transform $y(t) = \mathcal{F}^{-1}\{Y(j\omega)\}$.

Signal filtering – example (transfer f. approach)

$$x(t)=\operatorname{sgn}(t)\leftrightarrow\frac{2}{j\omega}$$

$$H(j\omega)=\Pi_{2W}(\omega)$$



$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\omega)\Pi_{2W}(\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-W}^{W}\frac{2}{j\omega}e^{j\omega t}d\omega=$$

$$=\frac{1}{\pi j}\int_{-W}^{W}\frac{e^{j\omega t}}{\omega}d\omega=\frac{1}{\pi j}\int_{-W}^{W}\frac{\cos\omega t+j\sin\omega t}{\omega}d\omega=$$

$$\frac{2}{\pi}\int_{0}^{W}\frac{\sin\omega t}{\omega}d\omega=\frac{2}{\pi}\int_{0}^{Wt}\frac{\sin\nu}{\nu}d\nu=\frac{2}{\pi}\operatorname{Si}(Wt)$$

Sinus integral properties

$$\mathrm{Si}(x) = \int_0^x \sin\tau/\tau \, d\tau$$

1. Sinus integral is an odd function

$$\mathrm{Si}(-x) = \int_0^{-x} \sin\tau/\tau \, d\tau = \left| \begin{matrix} \tau = -u \\ d\tau = -du \end{matrix} \right| = -\int_0^x \sin u/u \, du = -\mathrm{Si}(x)$$

2. Sinus integral in zero vicinity (x ≈ 0)

$$\mathrm{Si}(x \approx 0) \approx \int_0^x 1 d\tau = x, \sin\tau/\tau \approx 1$$

$$\mathrm{Si}(0) = \int_0^0 \sin\tau/\tau \, d\tau = 0$$

Sinus integral properties

$$\mathrm{Si}(x) = \int_0^x \sin\tau/\tau \, d\tau$$

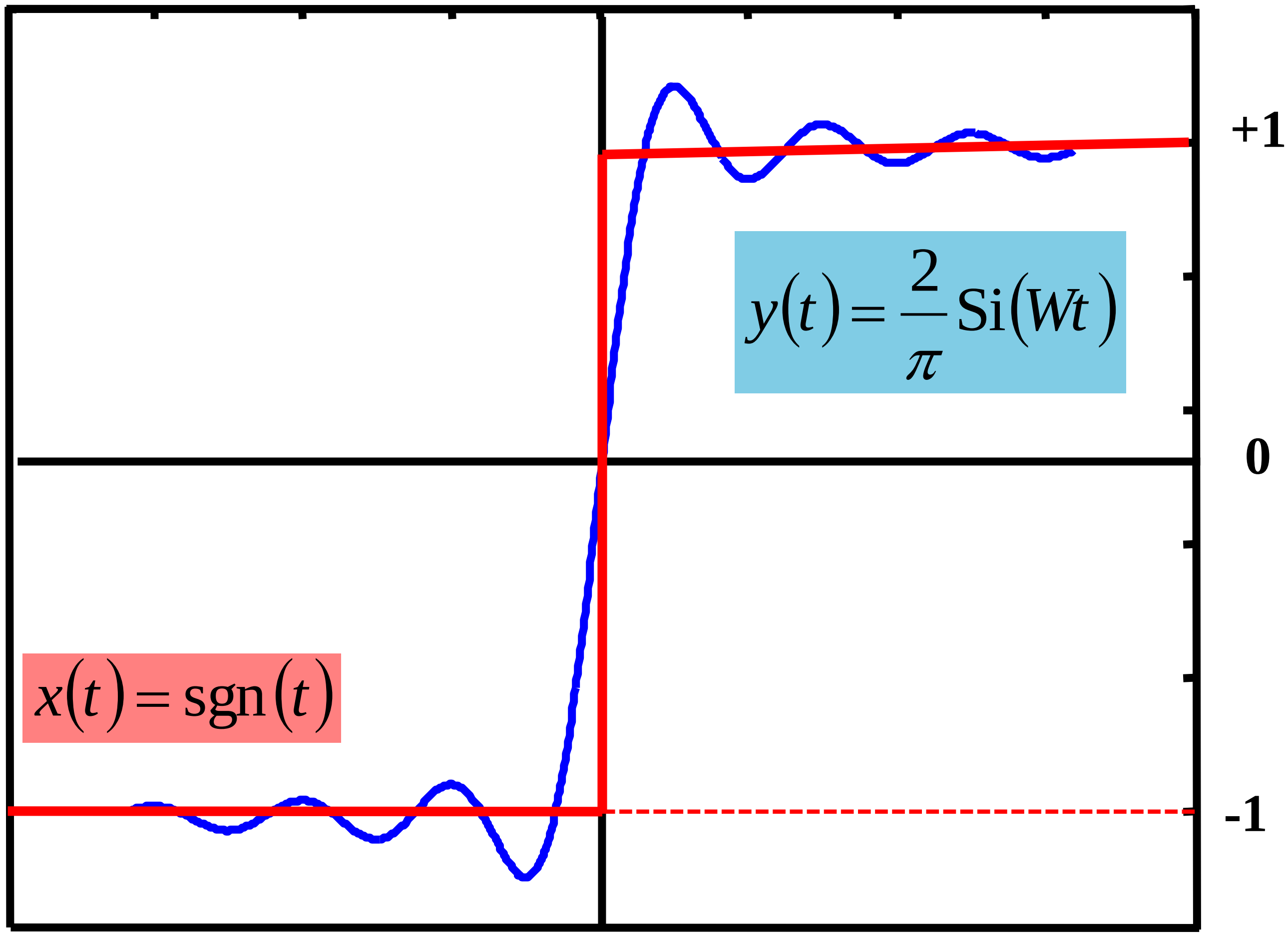
3. Horizontal asymptote (x →∞)

$$\lim_{x\to\infty} \mathrm{Si}(x) = \int_0^\infty \sin\tau/\tau \, d\tau = \pi/2$$

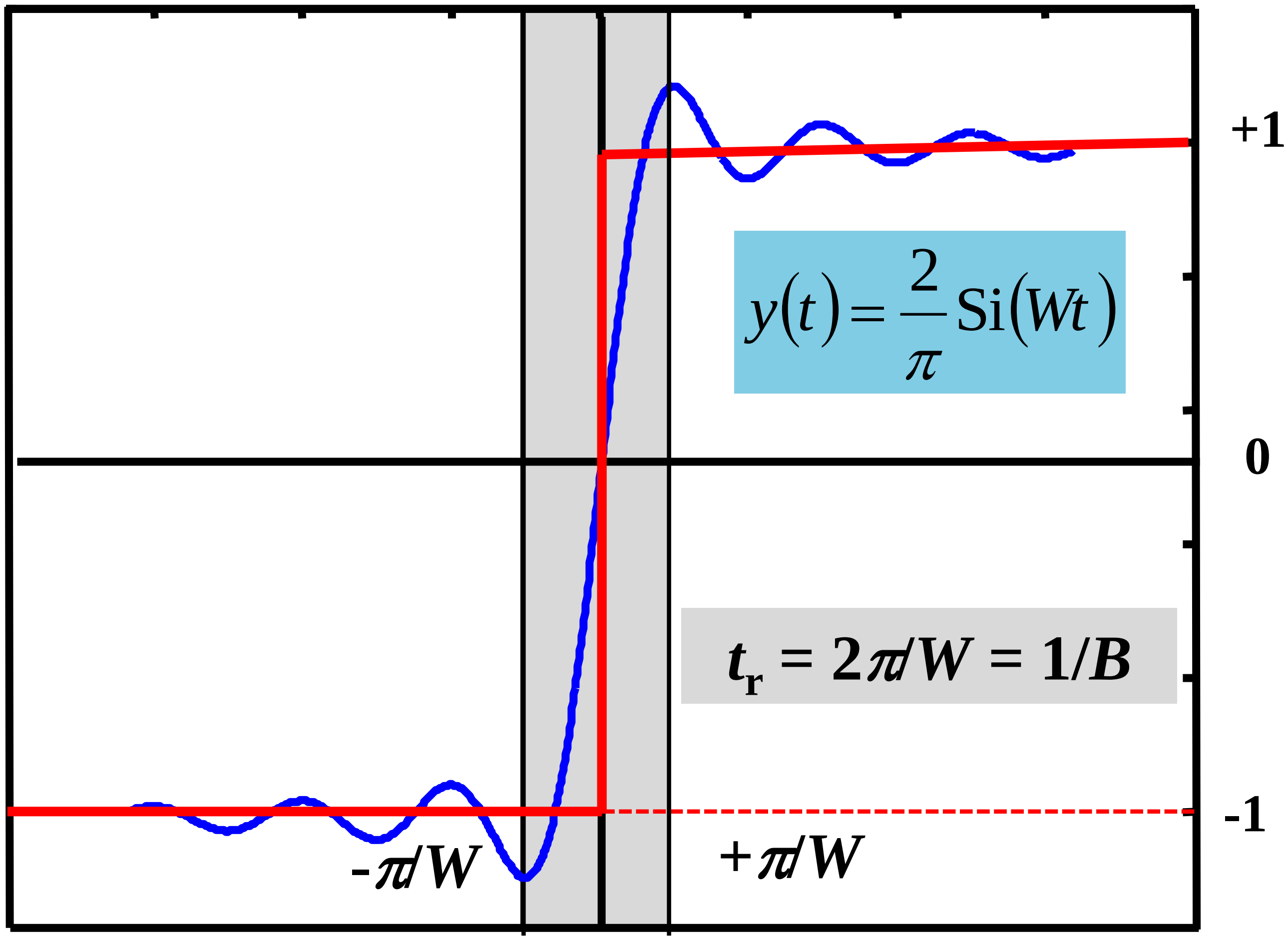
4. Local extrema

$$d\mathrm{Si}(x)/dx = \sin x/x = 0 \Leftrightarrow x = k\pi, k \neq 0$$

Lowpass filtering sgn(t) – output signal

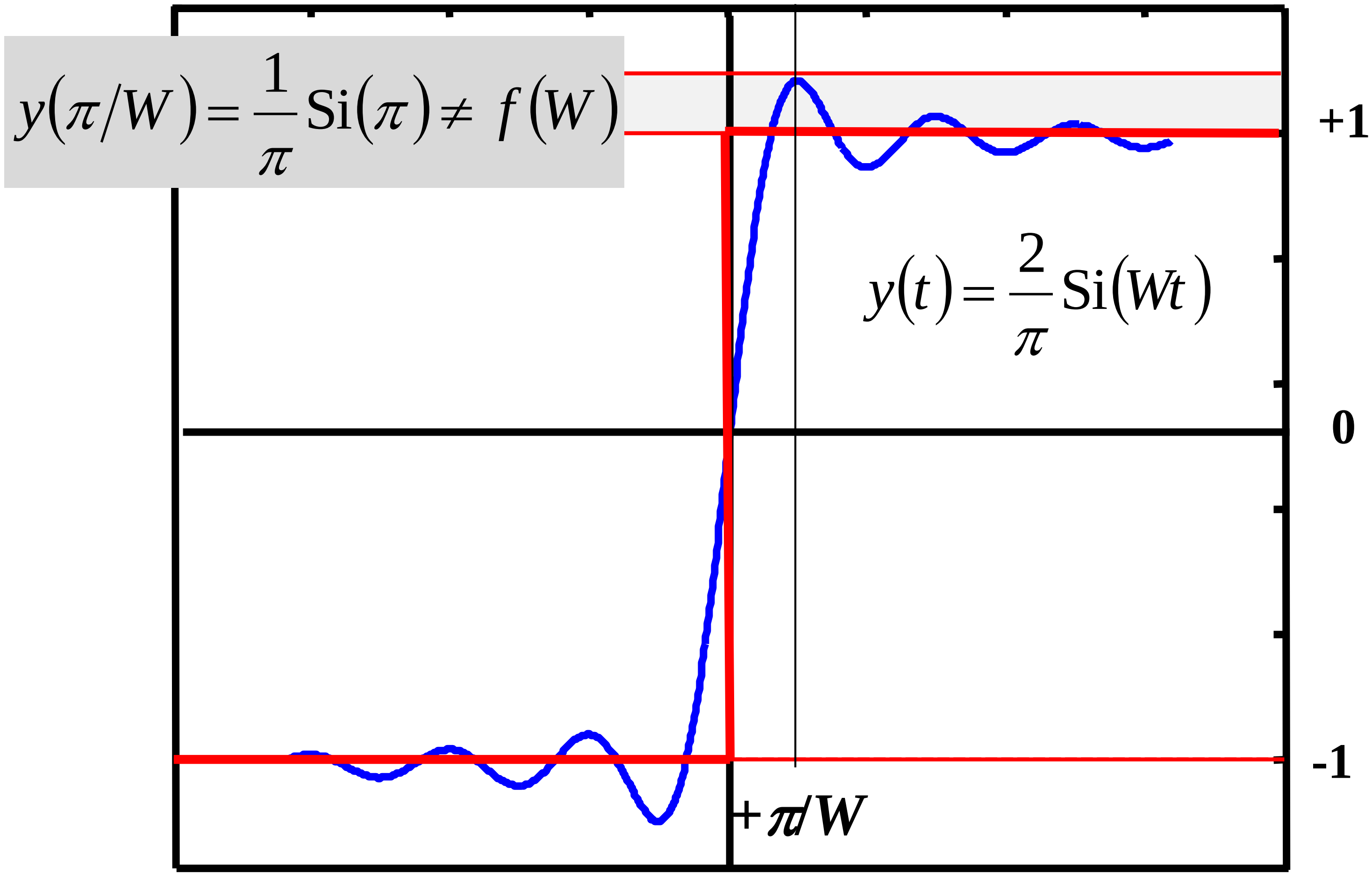


Lowpass filtering sgn(t) – slope rising time



Slope rising time is inversely proportional to a filter bandwidth.

Lowpass filtering sgn(t) – Gibbs effect



Peak value od the output signal is independent of the filter bandwidth (Gibbs effect).

Signal filtering - convolution

$$x(t) \leftrightarrow X(j\omega) \longrightarrow \boxed{H(s:=j\omega)} \longrightarrow y(t) \leftrightarrow Y(j\omega)$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$y(t) = h(t) * x(t)$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

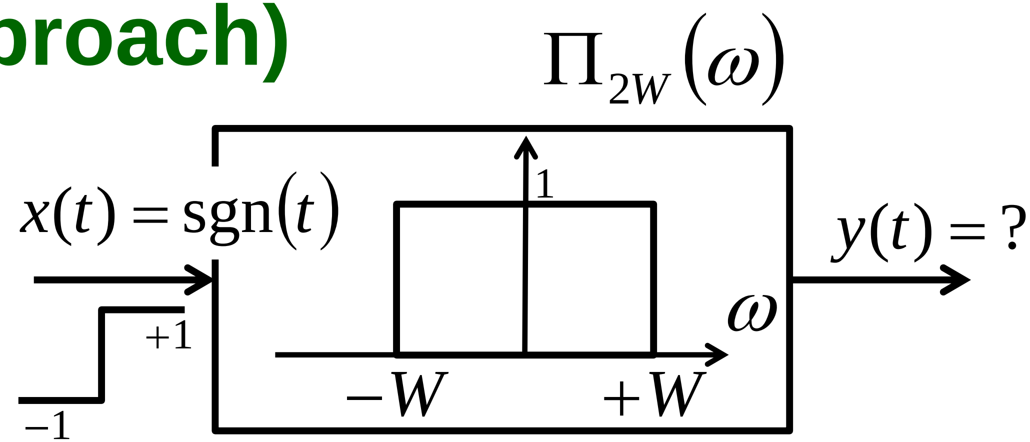
$h(t) \leftrightarrow H(j\omega)$

impulse response

of a filter

Output signal of a filter is a convolution
of input signal and an impulse response of a filter.

Signal filtering – example (convolution approach)



$$H(j\omega)=\Pi_{2W}(\omega)\leftrightarrow h(t)=\frac{W}{\pi}\mathrm{Sa}(Wt)$$

$$y(t)=\mathrm{sgn}(t)*\frac{W}{\pi}\mathrm{Sa}(Wt)=\frac{W}{\pi}\int_{-\infty}^{\infty}\mathrm{Sa}(W\tau)\mathrm{sgn}(t-\tau)d\tau=$$
$$\frac{W}{\pi}\int_{-\infty}^{t}\mathrm{Sa}(W\tau)d\tau-\frac{W}{\pi}\int_{t}^{+\infty}\mathrm{Sa}(W\tau)d\tau=\frac{2}{\pi}\mathrm{Si}(Wt)$$

Summary

The response of Time-Invariant Linear System (TILS) to a composite input is a sum of responses to individual input components.

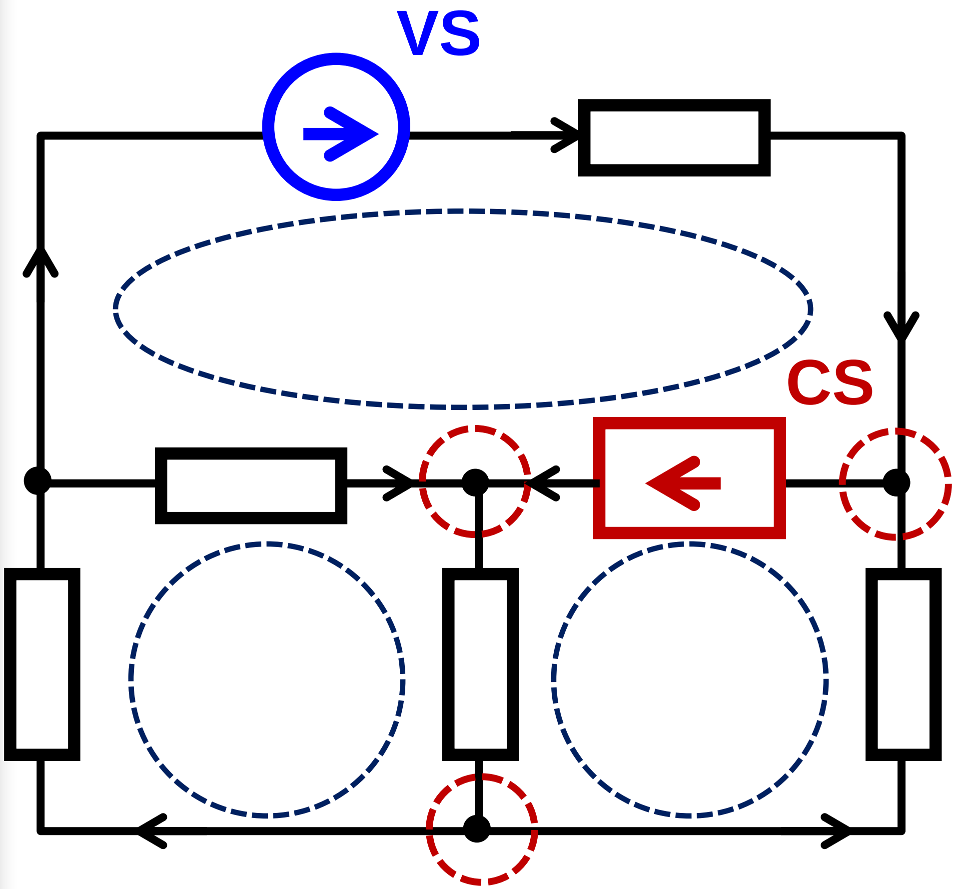
The exponential signal is an invariant to Linear Time-invariant Systems (TILS).

Fourier series and Fourier transform represent a signal through its spectrum which is a linear combination of exponential signals.

Fourier series is suited for periodic signals while Fourier transform is suited for unperiodic signals.

Signal filtering is described either in the frequency domain (multiplication of spectra) or in the time domain (convolution).

Kirchhoff laws in RLC networks



RLC network (example)

Kirchhoff Voltage Law

Kirchhoff Current Law

any RLC element

CS – current source of predefined profile *i*(*t*)

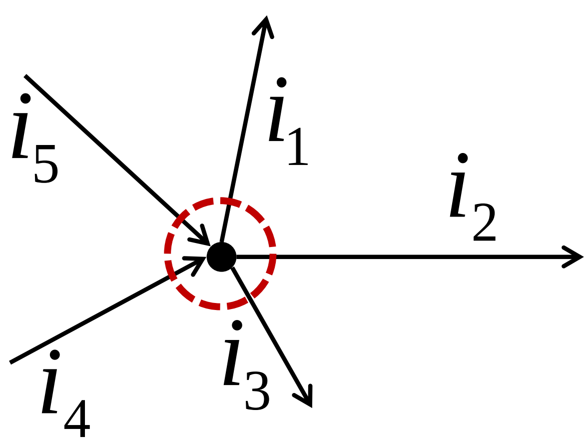
VS – voltage source of predefined profile *v*(*t*)

Kirchhoff Current Low (KCL)

The algebraic sum of currents in an electrical network meeting at a node is zero.

Note that current is a signed (positive or negative) quantity reflecting direction towards or away from a node.

The sum of currents flowing into that node is equal to the sum of currents flowing out of that node.



$$i_1 + i_2 + i_3 - i_4 - i_5 = 0$$

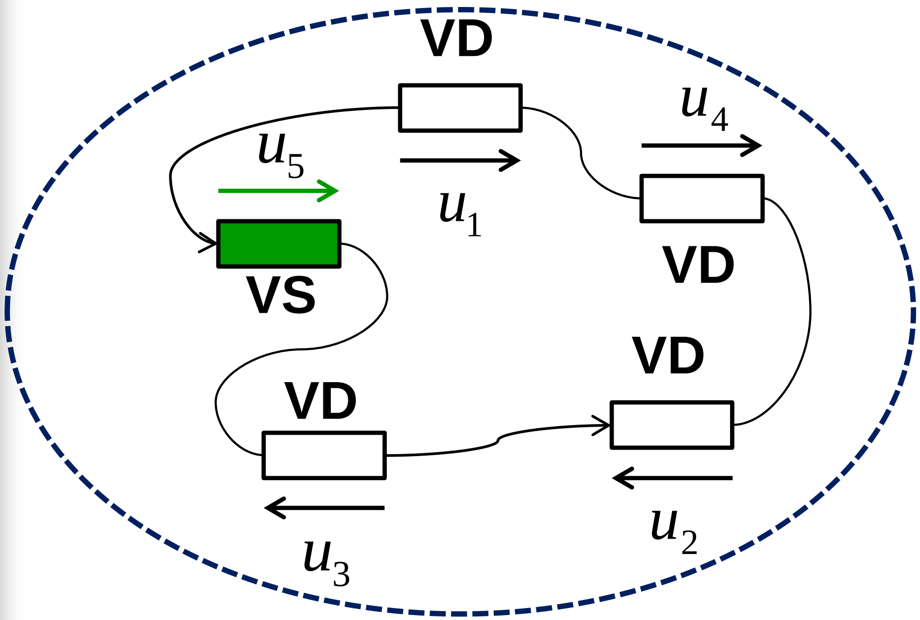
$$i_1 + i_2 + i_3 = i_4 + i_5$$

$$\sum_k i_k = 0$$

Kirchhoff Voltage Low (KVL)

The algebraic sum of voltage drops and voltage sources around any closed electrical loop is zero.

By agreement, voltage drop is a negative quantity while voltage source is a positive quantity.

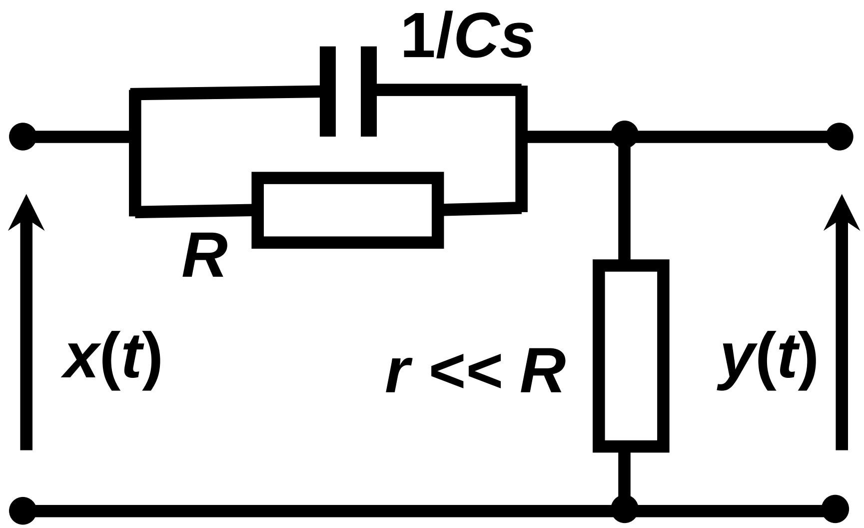


$$u_1 + u_2 + u_3 + u_4 - u_5 = 0$$

$$u_1 + u_2 + u_3 + u_4 = u_5$$

$$\sum_k u_k = 0$$

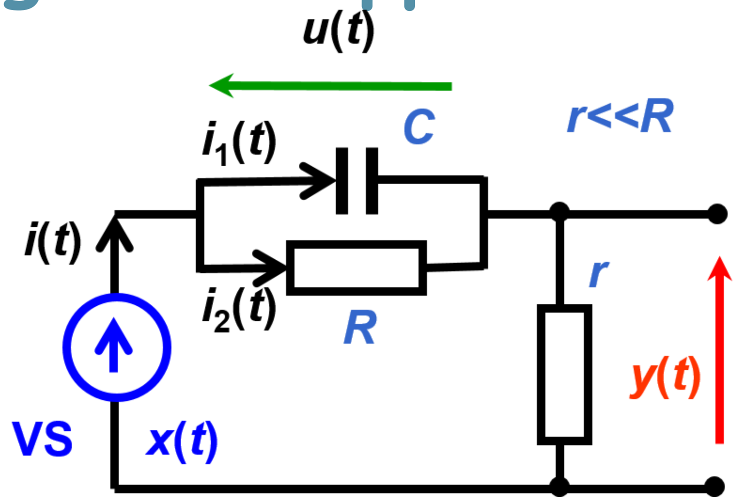
Preemphasis filter



A preemphasis filter at a transmitter (together with an accompanying deemphasis filter at a receiver) provide better noise immunity in both radio and television broadcasting.

$$H(s)=\frac{r}{r+\dfrac{R\times 1/Cs}{R+1/Cs}}\approx\frac{r}{R}\times\frac{1+RCs}{1+rCs}$$

Preemphasis filter – transfer function – generic approach



KVL KCL

$$x = u + y \qquad i = i_1 + i_2$$

V – C (RLC)

$$i_1 = Cu' \qquad i_2 = u/R \qquad y = ri$$

$$y = r\left(Cu' + \frac{u}{R}\right)$$

$$u = x - y$$

$$y = rC(x' - y') + \frac{r}{R}(x - y)$$

$$y\left(1 + \frac{r}{R}\right) + rCy' = rCx' + \frac{r}{R}x, \quad r/R \approx 0, \times R/r$$

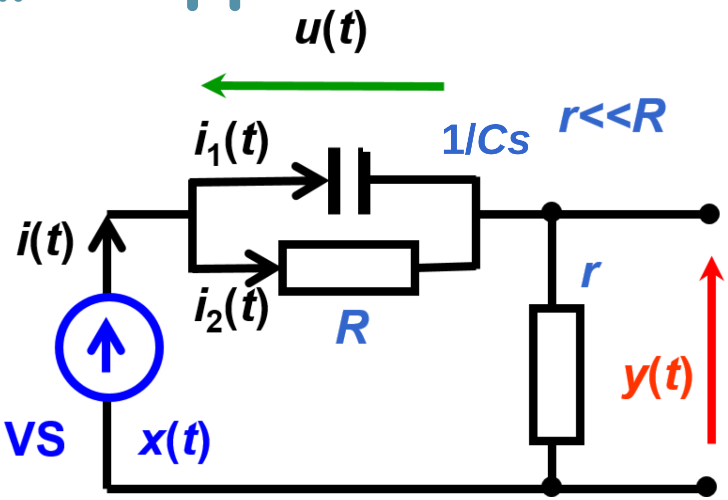
$$\frac{R}{r}(y + rCy') \approx RCx' + x$$

$$x = e^{st}, y(t) = H(s)e^{st}$$

$$H(s) \cong \frac{r}{R} \times \frac{1 + sRC}{1 + srC}$$

WORK IN PROGRESS

Preemphasis filter – transfer function – „s" approach



KVL KCL

$$x = u + y \quad i = i_1 + i_2$$

V – C (RLC)

$$i_1 = Csu \quad i_2 = u/R \quad y = ri$$

$$y = r\left(Csu + \frac{u}{R}\right)$$

$$u = x - y$$

$$y = rCs(x-y) + \frac{r}{R}(x-y)$$

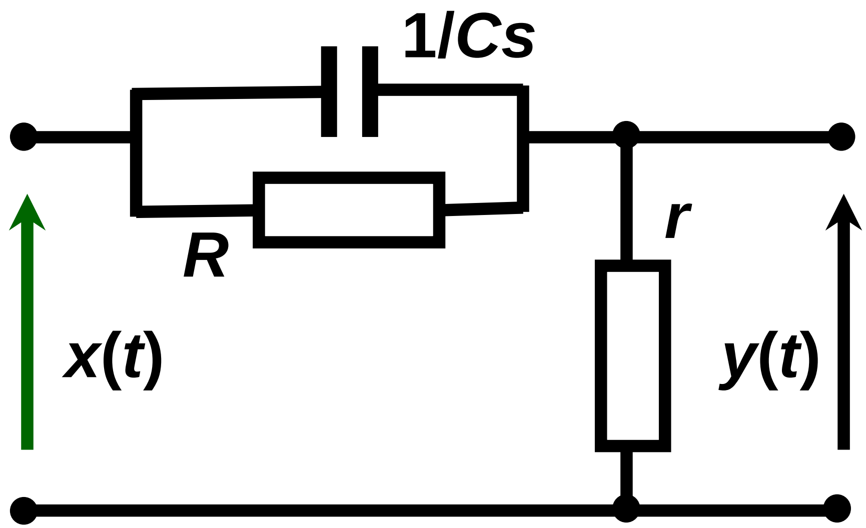
$$y\left(1 + \frac{r}{R} + rCs\right) = x\left(\frac{r}{R} + rCs\right), \quad r/R \approx 0$$

$$y(1 + rCs) \approx \frac{r}{R}x(1 + RCs)$$

$$x = e^{st}, y(t) = H(s)e^{st}$$

$$H(s) \cong \frac{r}{R} \times \frac{1 + sRC}{1 + srC}$$

Preemphasis filter – a-f characteristic



$$r \ll R$$

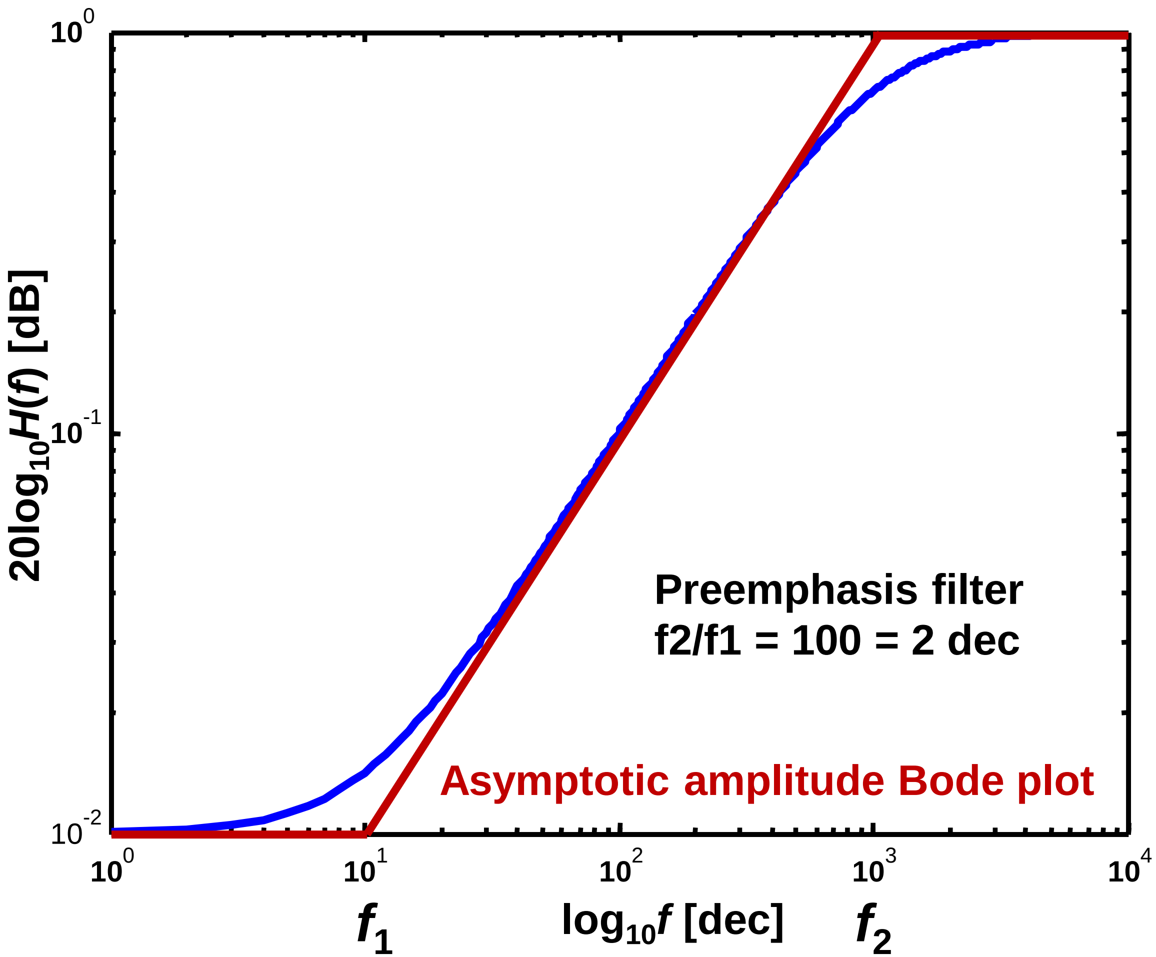
$$\omega_1 = \frac{1}{RC}, \omega_2 = \frac{1}{rC}$$

$$\omega_1 \ll \omega_2$$

$$H(j\omega) = \frac{r}{R} \times \left.\frac{1+sRC}{1+srC}\right|_{s=j\omega} = \frac{r}{R} \times \frac{1+j\omega/\omega_1}{1+j\omega/\omega_2}$$

$$A(\omega) = \left|H(j\omega)\right| = \frac{r}{R} \times \frac{\sqrt{1+(\omega/\omega_1)^2}}{\sqrt{1+(\omega/\omega_2)^2}} = \begin{cases} r/R, \omega \ll \omega_1 \\ \omega/\omega_2, \omega_1 \ll \omega \ll \omega_2 \\ 1, \omega \gg \omega_2 \end{cases}$$

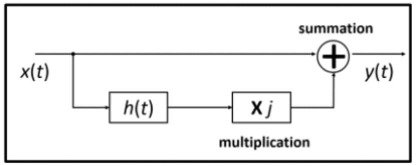
Preemphasis filter (a-f characteristic)



Log-log amplitude – frequency characteristic

Problems

The signal processing system is shown in the frame below.



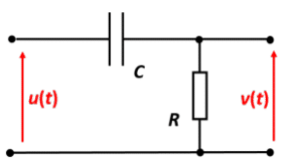
Q1. The impulse response of the filter is equal to $h(t) = \frac{1}{\pi t}$. Find the transfer function of the filter $H(j\omega)$ using the symmetry property.

Q2. Sketch both amplitude $A(\omega)$ and phase $\varphi(\omega)$ characteristics of the filter $H(j\omega)$ (in a linear coordinate system). Describe in your own words the operation of the filter

Q3. Find the Fourier transform $Y(\omega) \leftrightarrow y(t)$ of the output signal of the system and compare it with the Fourier transform of the input signal $X(\omega) \leftrightarrow x(t)$. How would you describe in words the operation of the signal processing system shown in the frame?

Q4. Let $x(t) = \cos \omega_0 t$. Find the output signal $y(t)$.

Q1. Find the voltage to voltage transfer function $H(s)$ of the unloaded *RC* voltage divider:



Q2. Find the amplitude-frequency characteristic $A(\omega)$ of the filter. Substitute for $(RC)^{-1} = \omega_b$.

Q3. Find the linear approximation of the characteristic and plot it in a linear coordinate system (both axes).

Q4. How would you describe in your own words the operation of the filter keeping in mind the Fourier transform property $x'(t) \leftrightarrow j\omega X(j\omega)$?